Squares by Matrices with Coherent Sequences (Infinitary combinatorics in set theory and its applications)

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Squares by Matrices with Coherent Sequences

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Abstract

We formulate a matrix with coherent sequences that entail squares. A matrix comprises models of set theory of a size equal to the least uncountable cardinal. A matrix with coherent sequences entail a simplified morass with linear limits. A simplified morass with linear limits entails squares by Velleman. Hence, a matrix with coherent sequences entails squares. We provide a direct proof of this fact. This study is based on Velleman's construction of squares by a simplified morass with linear limits.

Introduction

Velleman introduced simplified morasses as an alternative to constructions in the constructible universe ([V1], [V2], [V3]). Koszmider followed Velleman to formulate semimorasses ([K]). Todorcevic concieved matrices of isomorphic models of set theory along his so-called side condition methods ([T1], [T2]). Aspero and Mota rediscovered the use of matrices ([A-M]). Shelah and Baumgartner had a forcing construction in that each condition keeps its history ([B-S]). We noted a connection between these types of objects in the universe of set theory: namely, certain kinds of matrices of isomorphic models of set theory entail simplified morasses, semimorasses, and quagmires ([M1], [M2]). In this paper, we consider a matrix with coherent sequences that entails a simplified $(\omega_2, 1)$-morass with linear limits ([M3]). Simplified $(\omega_2, 1)$-morasses with linear limits entail $\square_{\omega_2}$ by Velleman. He provided two proofs of this implication. We sort of combine these two proofs to directly show that matrices with coherent sequences entail $\square_{\omega_2}$. This study is motivated by a question posed by Brooke-Taylor during my presentation on matrices of isomorphic models in the RIMS set theory workshop, Kyoto, 2013.

§1. A matrix with coherent sequences

We formulate a matrix with coherent sequences. Since we are not sure which direction to proceed in this line of study yet, our treatment of this subject tends to be rather ad hoc ([M1], [M2], [M3]).

1.1 Definition. Let H be a transitive set model of a sufficient fragment of set theory such that

- $\omega_3 \subset H \subset H_{\omega_3}$.
- ${}^{\omega_1}H \subset H$: namely for any sequence $f : \omega_1 \longrightarrow H$, we demand $f \in H$.

In particular, we have

- If $M \subset H$ with $|M| = \kappa \in \{\omega, \omega_1\}$, then $M \in H$ and $H \models$ "$|M| = \kappa$".
- ω_1, ω_2 are definable in H with no parameters and are absolute between H and H_{ω_3}.

Typically, H is H_{ω_3} in the ground model V and we are in the generic extensions $V[G]$, where G are P-generic over V, and P is a notion of forcing that forces a matrix with coherent sequences. We may assume that P is σ-closed, ω_2-Baire (no new sequences of ordinals of length ω_1 get created), and has the ω_3-c.c. under $2^{\omega_1} = \omega_2$ ([M3]).

Let $\mathcal{M}_1$ be a set of elementary substructures of a prefixed structure $(H, \in, \cdots)$ such that

- For each $M \in \mathcal{M}_1$, it is required that $|M| = \omega_1$ and $(\omega_1 <)\, M \cap \omega_2 < \omega_2$.
- $\mathcal{M}_1$ is closed under finite intersections: for $M, M' \in \mathcal{M}_1$, $M \cap M' \in \mathcal{M}_1$.
- $\mathcal{M}_1$ is closed under taking the unions of $\in$-increasing sequences of elements, at most of a length ω_1: if $\langle M_i \mid i < \nu \rangle$ is an $\in$-increasing sequence of elements of $\mathcal{M}_1$ with $\nu \leq \omega_1$, then $\bigcup\{M_i \mid i < \nu\} \in \mathcal{M}_1$.
- $\mathcal{M}_1$ is $\in$-cofinal in H: $\bigcup \mathcal{M}_1 = H$.

- If $M, M' \in \mathcal{M}_1$ and $\phi : (M, \in, \cdots) \longrightarrow (M', \in, \cdots)$ is an isomorphism such that ϕ is the identity on the intersection $M \cap M'$, then for any $M'' \in M \cap \mathcal{M}_1$, we demand $\phi(M'') \in \mathcal{M}_1$.

Typically, $\mathcal{M}_1$ comprises the elementary substructures M of $(H^V_{\omega_3}, \in, \lhd)$, where $\lhd$ well-orders $H^V_{\omega_3}$ in the ground model V, such that $|M| = \omega_1$ and $M \cap \omega_2 < \omega_2$ in V.

We record the following.

1.2 Proposition. Let $M, M' \in \mathcal{M}_1$.

(1) If $M \in M'$, then $M \subset M'$ (proper inclusion). In particular, $(\mathcal{M}_1, \in)$ is a well-founded strongly partially ordered set (irreflexive, transitive and has no infinitely $\in$-descending sequences).

(2) If $\phi : (M, \in, \cdots) \longrightarrow (M', \in, \cdots)$ is an isomorphism, then it is unique, $\phi(\omega_1) = \omega_1$, $\phi(\omega_2) = \omega_2$, and if $X \in M$ with $|X| = \omega_1$, we have $\phi(X) = \{\phi(x) \mid x \in X\}$, denoted by $\phi[X]$. In particular, if $X \in M$, then we have $\phi(X \cap \omega_1) = \phi[X \cap \omega_1] = \phi[X] \cap \omega_1 = \phi(X) \cap \omega_1$. If $X \in M$ with $|X| = \omega_1$, then we have

$$\phi(X \cap \omega_2) = \phi[X \cap \omega_2] = \phi[X] \cap \omega_2 = \phi(X) \cap \omega_2,$$

$$\phi(X \cap \omega_3) = \phi[X \cap \omega_3] = \phi[X] \cap \omega_3 = \phi(X) \cap \omega_3.$$

Prior to introducing homogeneity, we consider 4 types of so-called history $\mathcal{M} \cap M$ for each member $M \in \mathcal{M}$, where $\mathcal{M}$ is a given subset of $\mathcal{M}_1$.

1.3 Definition. Let $\mathcal{M} \subset \mathcal{M}_1$. Define

- $\mathrm{zero}(\mathcal{M}) = \{M \in \mathcal{M} \mid \mathcal{M} \cap M = \emptyset\}$.
- $\mathrm{suc}_1(\mathcal{M}) = \{M \in \mathcal{M} \mid$ there exists (unique) M_1 such that $\mathcal{M} \cap M = (\mathcal{M} \cap M_1) \cup \{M_1\}\}$.
- $\mathrm{suc}_2(\mathcal{M}) = \{M \in \mathcal{M} \mid$ there exist (unique) M_1, M_2 such that $M_1 \cap \omega_2 = M_2 \cap \omega_2$, $(M_1 \cap \omega_3) \cap (M_2 \cap \omega_3)$ is a proper initial segment of both $M_1 \cap \omega_3$ and $M_2 \cap \omega_3$, $M_1 \cap \omega_3 \subset \min\big((M_2 \cap \omega_3) \setminus M_1\big)$, and that $\mathcal{M} \cap M = (\mathcal{M} \cap M_1) \cup (\mathcal{M} \cap M_2) \cup \{M_1, M_2\}\}$.
- $\lim(\mathcal{M}) = \{M \in \mathcal{M} \mid \bigcup(\mathcal{M} \cap M) = M\}$.

We note that in $\lim(\mathcal{M})$, $\bigcup(\mathcal{M} \cap M) = M$ entails that $\mathcal{M} \cap M$ is $\in$-directed. We are interested in subsets $\mathcal{M}$ of $\mathcal{M}_1$ that are partitioned into the 4 parts:

$$\mathcal{M} = \mathrm{zero}(\mathcal{M}) \cup \mathrm{suc}_1(\mathcal{M}) \cup \mathrm{suc}_2(\mathcal{M}) \cup \lim(\mathcal{M}).$$

1.4 Defintion. $\mathcal{M}$ is called a *matrix* (of isomorphic models of set theory), if

(1) $\mathcal{M}$ is an $\in$-cofinal (equivalently, $\bigcup \mathcal{M} = H$) subset of $\mathcal{M}_1$.

(2) If $M, M' \in \mathcal{M}$ with $M \cap \omega_2 = M' \cap \omega_2$, then there exists an (unique) isomorphism $\phi : (M, \in, \cdots) \longrightarrow (M', \in, \cdots)$ such that ϕ is the identity on the intersection $M \cap M'$, and that $\phi[\mathcal{M} \cap M] = \mathcal{M} \cap M'$.

(3) If $\underline{M}, M' \in \mathcal{M}$ with $\underline{M} \cap \omega_2 < M' \cap \omega_2$, then there exists $M \in \mathcal{M}$ such that $\underline{M} \in M$ and $M \cap \omega_2 = M' \cap \omega_2$.

(4) $\mathcal{M}$ gets partitioned into the 4 parts.

Item (2) is called the homogeneity of $\mathcal{M}$. We may call item (3), upward-density of $\mathcal{M}$. Note that the $\in$-cofinal in (1) entails $\in$-directed: i.e. for each M, $M' \in \mathcal{M}$, there exists $M'' \in \mathcal{M}$ with $M, M' \in M''$. It is shown that if $\mathcal{M}$ is a matrix, then $I^{\mathcal{M}} = \{M \cap \omega_2 \mid M \in \mathcal{M}\}$ is a cub subset, consisting of limit ordinals, of ω_2, and that $\{M \cap \omega_3 \mid M \in \mathcal{M}\}$ forms a simplified $(\omega_2, 1)$-morass ([M3]).

1.5 Proposition. Let $\mathcal{M}$ be a matrix and $M, M' \in \mathcal{M}$. Then following are equivalent.

(1) The two $\in$-structures $(M, \in)$ and $(M', \in)$ are isomorphic.

(2) $M \cap \omega_2 = M' \cap \omega_2$.

(3) The two substructures $(M, \in, \cdots)$ and $(M', \in, \cdots)$ are isomorphic and the isomorphism is the identity on the intersection $M \cap M'$.

1.6 Proposition. Let $\mathcal{M}$ be a matrix and $M, M' \in \mathcal{M}$ be isomorphic with $\phi : M \longrightarrow M'$. Then ϕ preserves types of histories: namely

- $M \in \mathrm{zero}(\mathcal{M})$ iff $M' \in \mathrm{zero}(\mathcal{M})$.
- $M \in \mathrm{suc}_1(\mathcal{M})$ iff $M' \in \mathrm{suc}_1(\mathcal{M})$.
- $M \in \mathrm{suc}_2(\mathcal{M})$ iff $M' \in \mathrm{suc}_2(\mathcal{M})$.
- $M \in \lim(\mathcal{M})$ iff $M' \in \lim(\mathcal{M})$.

A matrix $\mathcal{M}$ is called a matrix *with coherent sequences*, if there exists a map $\langle M \mapsto \mathrm{LL}_M \mid M \in \lim(\mathcal{M})\rangle$ such that

- (linear) $\mathrm{LL}_M \subset \mathcal{M} \cap M$ and LL_M is well-ordered by $\in$.
- (cofinal) $\bigcup \mathrm{LL}_M = M$.
- (coherent) If $M' \in \mathrm{LL}_M$ such that $\mathrm{LL}_M \cap M'$ has no $\in$-last element, then $M' \in \lim(\mathcal{M})$ and $\mathrm{LL}_{M'} = \mathrm{LL}_M \cap M'$.
- (homogeneous) If $M, M' \in \lim(\mathcal{M})$ with the isomorphism $\phi : M \longrightarrow M'$, then $\phi[\mathrm{LL}_M] = \mathrm{LL}_{M'}$.
- (short) The order type of $(\mathrm{LL}_M, \in)$ is at most ω_1.

Hence, LL_M is a list of major events, so to speak, in the history $\mathcal{M} \cap M$ of the current stage M. We proved the following that is motivated by a question posed by Brooke-Taylor.

1.7 Theorem. ([M3]) (1) There exists a notion of forcing P that is σ-closed, ω_2-Baire, and has the ω_3-c.c. under $2^{\omega_1} = \omega_2$, and that there exists a matrix with coherent sequences in the generic extensions by P.

(2) If there exists a matrix with coherent sequences, then there exists a simplified $(\omega_2, 1)$-morass with linear limits.

Since simplified $(\omega_2, 1)$-morass with linear limits entails $\square_{\omega_2}$ ([V3]), so does a matrix with coherent sequences. We would like to provide a direct construction to this weaker implication.

§2. Squares by a matrix with coherent sequences

2.1 Theorem. If there exists a matrix with coherent sequences, then $\square_{\omega_1}$ and $\square_{\omega_2}$ hold.

It is rather straightforward to identify $\square_{\omega_1}$ out of $\langle \mathrm{LL}_M \mid M \in \lim(\mathcal{M})\rangle$: namely, $\{\underline{M} \cap \omega_2 \mid \underline{M} \in \mathrm{LL}_M\}$ provides a club at each $M \cap \omega_2$ with $M \in \lim(\mathcal{M})$, except that the whole space is $I^{\mathcal{M}}$ that is a club subset of ω_2. Now we concentrate on $\square_{\omega_2}$. We sort of combine two proofs found in [V3].

2.2 Definition. For each $M \in \mathcal{M}$, let

$$A^M = \{\sup(\underline{M} \cap \omega_3) \mid \underline{M} \in \mathcal{M} \cap M\}.$$

Hence, we are concentrating on one aspect $\sup(\cdot \cap \omega_3)$ of the history $\mathcal{M} \cap M$ of each M. We have

$$A^M \subset S^3_0 \cup S^3_1 = \{\xi < \omega_3 \mid \mathrm{cf}(\xi) = \omega\} \cup \{\xi < \omega_3 \mid \mathrm{cf}(\xi) = \omega_1\}.$$

Since $\mathcal{M}$ has the partition, we classify

- If $M \in \mathrm{zero}(\mathcal{M})$, then $A^M = \emptyset$.
- Let $M \in \mathrm{suc}_1(\mathcal{M})$ with $\mathcal{M} \cap M = (\mathcal{M} \cap M_1) \cup \{M_1\}$. Then $A^M = A^{M_1} \cup \{\pi_1\}$, where $\pi_1 = \sup(M_1 \cap \omega_3)$.
- Let $M \in \mathrm{suc}_2(\mathcal{M})$ with $\mathcal{M} \cap M = (\mathcal{M} \cap M_1) \cup (\mathcal{M} \cap M_2) \cup \{M_1, M_2\}$ and $\sup(M_1 \cap \omega_3) < \sup(M_2 \cap \omega_3)$. Then $A^M = A^{M_1} \cup A^{M_2} \cup \{\pi_1, \pi_2\}$, where $\pi_1 = \sup(M_1 \cap \omega_3)$, $\pi_2 = \sup(M_2 \cap \omega_3)$, and so $\pi_1 < \pi_2$.
- If $M \in \lim(\mathcal{M})$, then $A^M = \bigcup\{A^{\underline{M}} \mid \underline{M} \in \mathcal{M} \cap M\} = \bigcup\{A^{\underline{M}} \mid \underline{M} \in \mathrm{LL}_M\} = \bigcup\{A^{M_i} \mid i < \nu^M\}$, where $\langle M_i \mid i < \nu^M\rangle$ denotes the natural listing of LL_M.

In particular, abusively writting,

- If $M \in \mathrm{suc}_1(\mathcal{M})$, then $\max(A^M) = \pi_1$.
- If $M \in \mathrm{suc}_2(\mathcal{M})$, then $\max(A^M) = \pi_2$.
- If $M \in \lim(\mathcal{M})$, then there exists no last elements of A^M and the sequence $\langle \sup(M_i \cap \omega_3) \mid i < \nu^M \rangle$ is $<$-increasing countinuous, and cofinal in A^M.

  Therefore,

- $M \in \mathrm{zero}(\mathcal{M})$ iff $A^M = \emptyset$.
- $M \in \mathrm{suc}_1(\mathcal{M}) \cup \mathrm{suc}_2(\mathcal{M})$ iff $A^M \neq \emptyset$ has a max.
- $M \in \lim(\mathcal{M})$ iff $A^M \neq \emptyset$ has no last element.

  We have the homogeneity of A^M. Let $M, M' \in \mathcal{M}$ with the isomorphism $\phi : M \longrightarrow M'$. Then

$$\phi[A^M] = \{\phi(\sup(\underline{M} \cap \omega_3)) \mid \underline{M} \in \mathcal{M} \cap M\} = \{\sup(M'' \cap \omega_3) \mid M'' \in \mathcal{M} \cap M'\} = A^{M'}.$$

In particular,

- If $M \in \mathrm{suc}_1(\mathcal{M})$ with $\mathcal{M} \cap M = (\mathcal{M} \cap M_1) \cup \{M_1\}$, then $\mathcal{M} \cap M' = \mathcal{M} \cap \phi(M_1) \cup \{\phi(M_1)\}$ and $\phi(\sup(M_1 \cap \omega_3)) = \sup(\phi(M_1) \cap \omega_3)$.
- If $M \in \mathrm{suc}_2(\mathcal{M})$ with $\mathcal{M} \cap M = (\mathcal{M} \cap M_1) \cup (\mathcal{M} \cap M_2) \cup \{M_1, M_2\}$, then $\mathcal{M} \cap M' = (\mathcal{M} \cap \phi(M_1)) \cup (\mathcal{M} \cap \phi(M_2)) \cup \{\phi(M_1), \phi(M_2)\}$, $\phi(\sup(M_1 \cap \omega_3)) = \sup(\phi(M_1) \cap \omega_3))$, and $\phi(\sup(M_2 \cap \omega_3)) = \sup(\phi(M_2) \cap \omega_3))$.
- If $M \in \lim(\mathcal{M})$, then $\phi[\{\sup(\underline{M} \cap \omega_3) \mid \underline{M} \in \mathrm{LL}_M\}] = \{\sup(\underline{M}' \cap \omega_3) \mid \underline{M}' \in \mathrm{LL}_{M'}\}$.

2.3 Definition. We recursively construct F^M_τ ($\tau \in A^M$) such that

- $F^M_\tau \subseteq A^M \cap \tau$.
- $\mathrm{ssup}(F^M_\tau) = \mathrm{ssup}(A^M \cap \tau)$.
- For $\underline{M} \in \mathcal{M} \cap M$ with $\tau \in A^{\underline{M}}$, we demand $F^{\underline{M}}_\tau \subseteq_{\mathrm{end}} F^M_\tau$.
- For two isomorphic $M', M'' \in \mathcal{M}$ such that $M' \cap \omega_2 = M'' \cap \omega_2 < M \cap \omega_2$, we demand $\phi[F^{M'}_\tau] = F^{M''}_{\phi(\tau)}$ for all $\tau \in A^{M'}$, where $\phi : M' \longrightarrow M''$, the isomorphism.

Here for a set of ordinals X, $\mathrm{ssup}(X)$ denotes the strong-sup of X: namely, the least ordinal α such that $X \subseteq \alpha$. Let $A = \{\sup(M \cap \omega_3) \mid M \in \mathcal{M}\}$. Then we may think of F^M_τ as a record of $(A \cap \tau)$'s history $A^M \cap \tau$ in the current stage of M, in a partial but excellent manner.

Depending on which cell M belongs to and relative positions of τ in A^M, we make several specifications on F^M_τ.

- $M \in \mathrm{zero}(\mathcal{M})$: $A^M = \emptyset$. Hence, there exists no τ to set F^M_τ.
- $M \in \mathrm{suc}_1(\mathcal{M})$: Let $A^M = A^{M_1} \cup \{\pi_1\}$.

$$F^M_\tau = \begin{cases} \emptyset, & \text{if } \tau = \pi_1 \text{ and } M_1 \in \mathrm{zero}(\mathcal{M}). \\ \{\max(A^{M_1})\}, & \text{if } \tau = \pi_1 \text{ and } M_1 \in \mathrm{suc}_1(\mathcal{M}) \cup \mathrm{suc}_2(\mathcal{M}). \\ \{\sup(\underline{M} \cap \omega_3) \mid \underline{M} \in \mathrm{LL}_{M_1}\}, & \text{if } \tau = \pi_1 \text{ and } M_1 \in \lim(\mathcal{M}). \\ F^{M_1}_\tau, & \text{if } \tau \in A^{M_1}. \end{cases}$$

- $M \in \mathrm{suc}_2(\mathcal{M})$: Let $A^M = A^{M_1} \cup A^{M_2} \cup \{\pi_1, \pi_2\}$.

$$F^M_{\pi_2} = \begin{cases} \{\pi_1\}, & \text{if } M_2 \in \mathrm{zero}(\mathcal{M}). \\ \max\{\max(A^{M_2}), \pi_1\}, & \text{if } M_2 \in \mathrm{suc}_1(\mathcal{M}) \cup \mathrm{suc}_2(\mathcal{M}). \\ \{\sup(\underline{M} \cap \omega_3) \mid \underline{M} \in \mathrm{LL}_{M_2}\}, & \text{if } M_2 \in \lim(\mathcal{M}). \end{cases}$$

Let $\eta_2 = \min(A^{M_2} \setminus M_1)$ and $\eta_1 = \min(A^{M_1} \setminus M_2)$, if any. For $\tau \in A^{M_2}$,

$$F_\tau^M = \begin{cases} F_\tau^{M_2}, & \text{if } \eta_2 < \tau. \\ F_{\eta_2}^{M_2} \cup \{\pi_1\}, & \text{if } \tau = \eta_2. \\ F_\tau^{M_2}, & \text{if } \tau \in A^{M_1} \cap A^{M_2}. \end{cases}$$

$$F_{\pi_1}^M = \begin{cases} \emptyset, & \text{if } M_1 \in \mathrm{zero}(\mathcal{M}). \\ \{\max(A^{M_1})\}, & \text{if } M_1 \in \mathrm{suc}_1(\mathcal{M}) \cup \mathrm{suc}_2(\mathcal{M}). \\ \{\sup(\underline{M} \cap \omega_3) \mid \underline{M} \in \mathrm{LL}_{M_1}\}, & \text{if } M_1 \in \lim(\mathcal{M}). \end{cases}$$

For $\tau \in A^{M_1}(\backslash M_2)$, let

$$F_\tau^M = F_\tau^{M_1}.$$

- $M \in \lim(\mathcal{M})$: $A^M = \bigcup\{A^{\underline{M}} \mid \underline{M} \in \mathcal{M} \cap M\} = \bigcup\{A^{\underline{M}} \mid \underline{M} \in \mathrm{LL}_M\}$.
  For $\tau \in A^M$, let

$$F_\tau^M = \bigcup\{F_\tau^{\underline{M}} \mid \tau \in A^{\underline{M}},\ \underline{M} \in \mathcal{M} \cap M\} = \bigcup\{F_\tau^{\underline{M}} \mid \tau \in A^{\underline{M}},\ \underline{M} \in \mathrm{LL}_M\}.$$

The construction is straightforward by inductively showing that F_τ^M are homogeneous.

2.4 Claim. If $M, M' \in \mathcal{M}$ with the isomorphism $\phi : M \longrightarrow M'$, then for $\tau \in A^M$, we have $\phi[F_\tau^M] = F_{\phi(\tau)}^{M'}$.

Proof. By induction on $M \cap \omega_2$.

□

2.5 Lemma. Let $M \in \mathcal{M}$ and $\tau, \pi \in A^M$. Let γ be a limit ordinal with $\gamma \leq \tau < \pi$. If $\sup(F_\tau^M \cap \gamma) = \sup(F_\pi^M \cap \gamma) = \gamma$, then there exists $(\underline{M}, \underline{\tau}, \underline{\pi})$ such that

- $\underline{M} \in \mathcal{M} \cap M$,
- $\underline{\tau}, \underline{\pi} \in A^{\underline{M}}$,
- $\gamma \leq \underline{\tau} \leq \underline{\pi}$,
- $\{F_{\underline{\tau}}^{\underline{M}} \cap \gamma, F_{\underline{\pi}}^{\underline{M}} \cap \gamma\} = \{F_\tau^M \cap \gamma, F_\pi^M \cap \gamma\}$.

2.6 Corollary. Let $M \in \mathcal{M}$ and $\tau, \pi \in A^M$. Let γ be a limit ordinal with $\gamma \leq \tau < \pi$. If $\sup(F_\tau^M \cap \gamma) = \sup(F_\pi^M \cap \gamma) = \gamma$, then $F_\tau^M \cap \gamma = F_\pi^M \cap \gamma$.

Proof. Try to apply repeatedly the lemma above. As long as $\underline{\tau} < \underline{\pi}$, we may continue. Since there exists no infinite $\in$-descending sequences of M's, it must stop. Hence we have (M', τ', π') such that

- $M' \in \mathcal{M} \cap M$,
- $\tau', \pi' \in A^{M'}$,
- $\gamma \leq \tau' = \pi'$,
- $\{F_{\tau'}^{M'} \cap \gamma, F_{\pi'}^{M'} \cap \gamma\} = \{F_\tau^M \cap \gamma, F_\pi^M \cap \gamma\}$.

In particular, we have $F_\tau^M \cap \gamma = F_\pi^M \cap \gamma$.

□

Proof of 2.5 Lemma. By induction on $(\mathcal{M}, \in)$.

Case. $M \in \lim(\mathcal{M})$: Pick $\underline{M} \in \mathcal{M} \cap M$ with $\tau, \pi \in A^{\underline{M}}$. Then,

- $\tau < \mathrm{ssup}(A^{\underline{M}} \cap \pi) = \mathrm{ssup}(F_\pi^{\underline{M}})$,
- $F_\pi^{\underline{M}} \subseteq_{\mathrm{end}} F_\pi^M$.

Hence,

- $F_{\pi}^{\underline{M}} \cap (\tau+1) = F_{\pi}^{M} \cap (\tau+1)$.

And so,

- $F_{\pi}^{\underline{M}} \cap \gamma = F_{\pi}^{M} \cap \gamma$.

Then,

- $F_{\pi}^{M} \cap \gamma \subseteq A^{\underline{M}} \cap \tau$,
- $\gamma \leq \mathrm{ssup}(A^{\underline{M}} \cap \tau) = \mathrm{ssup}(F_{\tau}^{\underline{M}})$,
- $F_{\tau}^{\underline{M}} \subseteq_{\mathrm{end}} F_{\tau}^{M}$.

Hence $F_{\tau}^{\underline{M}} \cap \gamma = F_{\tau}^{M} \cap \gamma$.

Case. $M \in \mathrm{suc}_1(\mathcal{M})$: Let $\underline{M} \in \mathcal{M} \cap M$ with $\mathcal{M} \cap M = (\mathcal{M} \cap \underline{M}) \cup \{\underline{M}\}$. We have $A^M = A^{\underline{M}} \cup \{\pi_1\}$, where $\pi_1 = \sup(\underline{M} \cap \omega_3)$.

Subcase 1. $\pi \in A^{\underline{M}}$: Then $\tau \in A^{\underline{M}}$. By definition, $F_{\pi}^{M} = F_{\pi}^{\underline{M}}$ and $F_{\tau}^{M} = F_{\tau}^{\underline{M}}$. Let $\underline{\tau} = \tau$ and $\underline{\pi} = \pi$. Then,

- $\gamma \leq \underline{\tau} < \underline{\pi}$,
- $F_{\underline{\pi}}^{\underline{M}} \cap \gamma = F_{\pi}^{M} \cap \gamma$,
- $F_{\underline{\tau}}^{\underline{M}} \cap \gamma = F_{\tau}^{M} \cap \gamma$.

Subcase 2. $\pi = \pi_1$: Then $\tau \in A^{\underline{M}}$ and $F_{\tau}^{\underline{M}} = F_{\tau}^{M}$. Since F_{π}^{M} is infinite, we have

- $\underline{M} \in \lim(\mathcal{M})$,
- $\gamma \in A^{\underline{M}}$,
- $F_{\gamma}^{\underline{M}} = F_{\gamma}^{M} = F_{\pi}^{M} \cap \gamma$.

Let $\underline{\tau} = \gamma$ and $\underline{\pi} = \tau$. Then,

- $\gamma = \underline{\tau} \leq \underline{\pi}$,
- $F_{\underline{\pi}}^{\underline{M}} \cap \gamma = F_{\tau}^{\underline{M}} \cap \gamma = F_{\tau}^{M} \cap \gamma$,
- $F_{\underline{\tau}}^{\underline{M}} \cap \gamma = F_{\gamma}^{\underline{M}} \cap \gamma = F_{\pi}^{M} \cap \gamma$.

Case. $M \in \mathrm{suc}_2(\mathcal{M})$: Let $M_1, M_2 \in \mathcal{M} \cap M$ such that $\mathcal{M} \cap M = (\mathcal{M} \cap M_1) \cup (\mathcal{M} \cap M_2) \cup \{M_1, M_2\}$. Let $\pi_1 = \sup(M_1 \cap \omega_3)$ and $\pi_2 = \sup(M_2 \cap \omega_3)$. We have $A^M = A^{M_1} \cup A^{M_2} \cup \{\pi_1, \pi_2\}$. Let η_2 be the least element of $A^{M_2} \setminus M_1$, if any, and η_1 be the least element of $A^{M_1} \setminus M_2$, if any. We have $(M_1, A^{M_1}, \eta_1) \approx (M_2, A^{M_2}, \eta_2)$. We have a dozen of subcases.

Subcase 1. $\pi = \pi_2$, $\tau \in A^{M_2}$ and $\eta_2 < \tau$. Then,

- $M_2 \in \lim(\mathcal{M})$,
- $\gamma \in A^{M_2}$,
- $F_{\pi}^{M} \cap \gamma = F_{\gamma}^{M_2}$.

By definition,

- $F_{\tau}^{M} = F_{\tau}^{M_2}$.

Let $\underline{M} = M_2$, $\underline{\tau} = \gamma$ and $\underline{\pi} = \tau$. Then,

- $\gamma = \underline{\tau} \leq \underline{\pi}$,
- $F_{\underline{\pi}}^{\underline{M}} \cap \gamma = F_{\tau}^{M_2} \cap \gamma = F_{\tau}^{M} \cap \gamma$,
- $F_{\underline{\tau}}^{\underline{M}} \cap \gamma = F_{\gamma}^{M_2} \cap \gamma = F_{\gamma}^{M_2} = F_{\pi}^{M} \cap \gamma$.

Subcase 2. $\pi = \pi_2$ and $\tau = \eta_2$: Then we have

- $M_2 \in \lim(\mathcal{M})$,
- $\gamma \in A^{M_2}$,
- $F^M_\pi \cap \gamma = F^{M_2}_\gamma$.

By definition, $F^M_\tau = F^{M_2}_\tau \cup \{\pi_1\}$. But $\gamma < \pi_1$. Hence,

- $F^M_\tau \cap \gamma = F^{M_2}_\tau \cap \gamma$.

Let $\underline{M} = M_2$, $\underline{\tau} = \gamma$ and $\underline{\pi} = \tau$. Then we have

- $\gamma = \underline{\tau} < \underline{\pi}$,
- $F^{\underline{M}}_{\underline{\pi}} \cap \gamma = F^{M_2}_\tau \cap \gamma = F^M_\tau \cap \gamma$,
- $F^{\underline{M}}_{\underline{\tau}} \cap \gamma = F^{M_2}_\gamma \cap \gamma = F^{M_2}_\gamma = F^M_\pi \cap \gamma$.

Subcase 3. $\pi = \pi_2$ and $\tau = \pi_1$: Then we have

- $M_2 \in \lim(\mathcal{M})$,
- $M_1 \approx M_2$,
- $\gamma \in A^{M_1} \cap A^{M_2}$,
- $F^M_\pi \cap \gamma = F^{M_2}_\gamma$,
- $F^M_\tau \cap \gamma = F^{M_1}_\gamma$,
- $F^{M_2}_\gamma = F^{M_1}_\gamma$.

Let $\underline{M} = M_1$ and $\underline{\tau} = \underline{\pi} = \gamma$. Then we have

- $\gamma = \underline{\tau} = \underline{\pi}$,
- $F^{\underline{M}}_{\underline{\pi}} \cap \gamma = F^{M_1}_\gamma \cap \gamma = F^{M_1}_\gamma = F^M_\tau \cap \gamma$,
- $F^{\underline{M}}_{\underline{\tau}} \cap \gamma = F^{M_1}_\gamma = F^{M_2}_\gamma = F^M_\pi \cap \gamma$.

Subcase 4. $\pi = \pi_2$ and $\tau \in A^{M_1}$: Then we have

- $M_2 \in \lim(\mathcal{M})$,
- $\gamma \in A^{M_1} \cap A^{M_2}$,
- $F^M_\pi \cap \gamma = F^{M_2}_\gamma = F^{M_1}_\gamma$,
- $F^M_\tau = F^{M_1}_\tau$.

Let $\underline{M} = M_1$, $\underline{\tau} = \gamma$ and $\underline{\pi} = \tau$. Then we have

- $\gamma = \underline{\tau} \leq \underline{\pi}$,
- $F^{\underline{M}}_{\underline{\pi}} \cap \gamma = F^{M_1}_\tau \cap \gamma = F^M_\tau \cap \gamma$,
- $F^{\underline{M}}_{\underline{\tau}} \cap \gamma = F^{M_1}_\gamma \cap \gamma = F^{M_1}_\gamma = F^M_\pi \cap \gamma$.

Subcase 5. $\pi \in A^{M_2}$, $\tau \in A^{M_2}$ and $\eta_2 < \tau$: By definition, $F^M_\pi = F^{M_2}_\pi$ and $F^M_\tau = F^{M_2}_\tau$. Let $\underline{M} = M_2$, $\underline{\tau} = \tau$ and $\underline{\pi} = \pi$. Then we have

- $\gamma \leq \underline{\tau} < \underline{\pi}$,
- $F^{\underline{M}}_{\underline{\pi}} \cap \gamma = F^{M_2}_\pi \cap \gamma = F^M_\pi \cap \gamma$,
- $F^{\underline{M}}_{\underline{\tau}} \cap \gamma = F^{M_2}_\tau \cap \gamma = F^M_\tau \cap \gamma$.

Subcase 6. $\pi \in A^{M_2}$, $\eta_2 < \pi$ and $\tau = \eta_2$: By definition,

- $F^M_\pi = F^{M_2}_\pi$,

- $F^M_\tau = F^{M_2}_\tau \cup \{\pi_1\}$.

  We also have

- $\gamma < \pi_1$.

  Hence we have $F^M_\tau \cap \gamma = F^{M_2}_\tau \cap \gamma$. Let $\underline{M} = M_2$, $\underline{\tau} = \tau$ and $\underline{\pi} = \pi$. Then we have

- $\gamma < \underline{\tau} < \underline{\pi}$,
- $F^{\underline{M}}_{\underline{\pi}} \cap \gamma = F^{M_2}_\pi \cap \gamma = F^M_\pi \cap \gamma$,
- $F^{\underline{M}}_{\underline{\tau}} \cap \gamma = F^{M_2}_\tau \cap \gamma = F^M_\tau \cap \gamma$.

Subcase 7. $\pi \in A^{M_2}$, $\eta_2 < \pi$ and $\tau = \pi_1$: By definition, $F^M_\pi = F^{M_2}_\pi$. We also have

- $M_1 \in \lim(\mathcal{M})$,
- $\gamma \in A^{M_1} \cap A^{M_2}$,
- $F^M_\tau \cap \gamma = F^M_\gamma = F^{M_1}_\gamma = F^{M_2}_\gamma$.

  Let $\underline{M} = M_2$, $\underline{\tau} = \gamma$ and $\underline{\pi} = \pi$. Then we have

- $\gamma = \underline{\tau} < \underline{\pi}$,
- $F^{\underline{M}}_{\underline{\pi}} \cap \gamma = F^{M_2}_\pi \cap \gamma = F^M_\pi \cap \gamma$,
- $F^{\underline{M}}_{\underline{\tau}} \cap \gamma = F^{M_2}_\gamma \cap \gamma = F^{M_2}_\gamma = F^M_\tau \cap \gamma$.

Subcase 8. $\pi \in A^{M_2}$, $\eta_2 < \pi$ and $\tau \in A^{M_1} \setminus M_2$: Let π' be the M_1-copy of π. Then, we have

- $F^{M_1}_{\pi'} \cap \gamma = F^{M_2}_\pi \cap \gamma = F^M_\pi \cap \gamma$.

  Hence, we have

- $\gamma < \tau, \pi'$,
- $\{F^{M_1}_\tau \cap \gamma, F^{M_1}_{\pi'} \cap \gamma\} = \{F^M_\tau \cap \gamma, F^M_\pi \cap \gamma\}$.

  Let $\underline{M} = M_1$, $\underline{\tau} = \min\{\tau, \pi'\}$ and $\underline{\pi} = \max\{\tau, \pi'\}$. Then, we have

- $\gamma < \underline{\tau} \leq \underline{\pi}$,
- $\{F^{\underline{M}}_{\underline{\tau}} \cap \gamma, F^{\underline{M}}_{\underline{\pi}} \cap \gamma\} = \{F^{M_1}_\tau \cap \gamma, F^{M_1}_{\pi'} \cap \gamma\}$.

Subcase 9. $\pi \in A^{M_2}$, $\eta_2 < \pi$ and $\tau \in A^{M_1} \cap A^{M_2}$: By definition, $F^M_\pi = F^{M_2}_\pi$ and $F^M_\tau = F^{M_1}_\tau = F^{M_2}_\tau$. Let $\underline{M} = M_2$, $\underline{\tau} = \tau$ and $\underline{\pi} = \pi$. Then we have

- $\gamma \leq \underline{\tau} < \underline{\pi}$,
- $F^{\underline{M}}_{\underline{\pi}} \cap \gamma = F^{M_2}_\pi \cap \gamma = F^M_\pi \cap \gamma$,
- $F^{\underline{M}}_{\underline{\tau}} \cap \gamma = F^{M_2}_\tau \cap \gamma = F^M_\tau \cap \gamma$.

Subcase 10. $\pi = \eta_2$ and $\tau = \pi_1$: By definition, $F^M_\pi = F^{M_2}_\pi \cup \{\pi_1\}$. We have

- $M_1 \in \lim(\mathcal{M})$,
- $\gamma \in A^{M_1} \cap A^{M_2}$,
- $F^M_\tau \cap \gamma = F^{M_1}_\gamma = F^{M_2}_\gamma$.

  Let $\underline{M} = M_2$, $\underline{\tau} = \gamma$ and $\underline{\pi} = \pi$. Then we have

- $\gamma = \underline{\tau} < \underline{\pi}$,
- $F^{\underline{M}}_{\underline{\pi}} \cap \gamma = F^{M_2}_\pi \cap \gamma = F^M_\pi \cap \gamma$,
- $F^{\underline{M}}_{\underline{\tau}} \cap \gamma = F^{M_2}_\gamma \cap \gamma = F^{M_2}_\gamma = F^M_\tau \cap \gamma$.

Subcase 11. $\pi = \eta_2$ and $\tau \in A^{M_1} \setminus M_2$: Then, we have

- $F^M_{\eta_1} = F^{M_1}_{\eta_1} = F^{M_2}_{\eta_2}$,
- $F^M_\pi = F^{M_2}_{\eta_2} \cup \{\pi_1\}$,
- $\gamma < \pi_1$,
- $F^M_\tau = F^{M_1}_\tau$.

Let $\underline{M} = M_1$, $\underline{\tau} = \eta_1$ and $\underline{\pi} = \tau$. Then, we have

- $\gamma < \underline{\tau} \leq \underline{\pi}$,
- $F^{\underline{M}}_{\underline{\pi}} \cap \gamma = F^{M_1}_\tau \cap \gamma = F^M_\tau \cap \gamma$,
- $F^{\underline{M}}_{\underline{\tau}} \cap \gamma = F^{M_1}_{\eta_1} \cap \gamma = F^{M_2}_{\eta_2} \cap \gamma = F^M_\pi \cap \gamma$.

Subcase 12. $\pi = \eta_2$ and $\tau \in A^{M_1} \cap A^{M_2}$: Then, we have

- $F^M_\pi = F^{M_2}_\pi \cup \{\pi_1\}$.
- $F^M_\tau = F^{M_1}_\tau = F^{M_2}_\tau$.
- $\gamma < \pi_1$.

Let $\underline{M} = M_2$, $\underline{\tau} = \tau$ and $\underline{\pi} = \pi$. Then, we have

- $F^{\underline{M}}_{\underline{\pi}} \cap \gamma = F^{M_2}_\pi \cap \gamma = F^M_\pi \cap \gamma$.
- $F^{\underline{M}}_{\underline{\tau}} \cap \gamma = F^{M_2}_\tau \cap \gamma = F^M_\tau \cap \gamma$.

Subcase 13. $\pi = \pi_1$: Then $\tau \in A^{M_1}$. We have

- $M_1 \in \lim(\mathcal{M})$,
- $\gamma \in A^{M_1}$,
- $F^M_\pi \cap \gamma = F^M_\gamma = F^{M_1}_\gamma$,
- $F^M_\tau = F^{M_1}_\tau$.

Let $\underline{M} = M_1$, $\underline{\tau} = \gamma$ and $\underline{\pi} = \tau$. Then, we have

- $\gamma = \underline{\tau} \leq \underline{\pi}$,
- $F^{\underline{M}}_{\underline{\pi}} \cap \gamma = F^{M_1}_\tau \cap \gamma = F^M_\tau \cap \gamma$,
- $F^{\underline{M}}_{\underline{\tau}} \cap \gamma = F^{M_1}_\gamma \cap \gamma = F^{M_1}_\gamma = F^M_\pi \cap \gamma$.

Subcase 14. $\pi \in A^{M_1}$: Then $\tau \in A^{M_1}$. By definition, $F^M_\pi = F^{M_1}_\pi$ and $F^M_\tau = F^{M_1}_\tau$. Let $\underline{M} = M_1$, $\underline{\tau} = \tau$ and $\underline{\pi} = \pi$. Then, we have

- $\gamma \leq \underline{\tau} < \underline{\pi}$,
- $F^{\underline{M}}_{\underline{\pi}} \cap \gamma = F^{M_1}_\pi \cap \gamma = F^M_\pi \cap \gamma$,
- $F^{\underline{M}}_{\underline{\tau}} \cap \gamma = F^{M_1}_\tau \cap \gamma = F^M_\tau \cap \gamma$.

$\square$

We continue our investigation of A^M's and F^M_τ's to show $\square_{\omega_2}$.

2.7 Definition. Let $A = \{\sup(M \cap \omega_3) \mid M \in \mathcal{M}\}$. For each $\tau \in A$, let $F_\tau = \bigcup\{F^M_\tau \mid \tau \in A^M, M \in \mathcal{M}\}$.

It is clear that $A = \bigcup\{A^M \mid M \in \mathcal{M}\}$ and that $\mathrm{o.t.}(A) = \omega_3$.

Claim 1. Let $\tau \in A$. Then we have

(1) If $M \in \mathcal{M}$ and $\tau \in A^M$, then $F^M_\tau \subseteq_{\mathrm{end}} F_\tau$.
(2) $\mathrm{ssup}(F_\tau) = \mathrm{ssup}(A \cap \tau)$.

(3) o.t.$(F_\tau) \leq \omega_2$.

(4) If o.t.$(F_\tau) = \omega_2$, then $A \cap \tau$ is a bounded subset of τ and cf(o.t.$(A \cap \tau)) = \omega_2$.

Claim 2. Let $\tau, \pi \in A$ and γ be a limit ordinal with $\gamma \leq \tau < \pi$. If $\sup(F_\pi \cap \gamma) = \sup(F_\tau \cap \gamma) = \gamma$, then $F_\pi \cap \gamma = F_\tau \cap \gamma$.

2.8 Definition. Let $\Phi : A \longrightarrow \omega_3$ be the transitive collapse. For each limit ordinal $i < \omega_3$ with, say, $\Phi(\pi) = i$, let

$$C_i = \overline{\Phi[F_\pi]} \setminus \{i\},$$

where $\overline{X}$ denote the closure of X in ω_3.

We know o.t.$(X) =$ o.t.$\big(\overline{X} \setminus \{\sup(X)\}\big)$ for X with no last elements. The sequence of these clubs C_i's satisfies $\square_{\omega_2}$.

Claim 3. Let i be a limit ordinal with $i < \omega_3$. Then we have

(1) C_i is closed and cofinal subset of i with o.t.$(C_i) \leq \omega_2$.

(2) If cf$(i) < \omega_2$, then o.t.$(C_i) < \omega_2$.

(3) If j is a limit ordinal with $j \in \overline{C_i} \cap i$, then $C_j = C_i \cap j$.

Proof of Claim 1. For (1): Let $y < x$ with $y \in F_\tau$ and $x \in F_\tau^M$. Pick $M' \in \mathcal{M}$ with $y \in F_\tau^{M'}$. Pick $M'' \in \mathcal{M}$ with $M, M' \in M''$. Then we have $y \in F_\tau^{M''}$ and $F_\tau^M \subseteq_{\text{end}} F_\tau^{M''}$. Hence, $y \in F_\tau^M$.

For (2): Since $F_\tau \subseteq A \cap \tau$, we have $\text{ssup}(F_\tau) \leq \text{ssup}(A \cap \tau)$. To show the converse, let $x \in A \cap \tau$. Pick $M \in \mathcal{M}$ with $\tau \in A^M$ and $x \in A^M \cap \tau$. Then $x < \text{ssup}(A^M \cap \tau) = \text{ssup}(F_\tau^M) \leq \text{ssup}(F_\tau)$. Hence $\text{ssup}(A \cap \tau) \leq \text{ssup}(F_\tau)$.

For (3): We have o.t.$(F_\tau^M) \leq$ o.t.$(A^M \cap \tau) <$ o.t.$(M \cap \omega_3) < \omega_2$ and $F_\tau^M \subseteq_{\text{end}} F_\tau$. Hence, o.t.$(F_\tau) \leq \omega_2$.

For (4): Suppose o.t.$(F_\tau) = \omega_2$. Then F_τ has no last elmenets and is a cofinal subset of $A \cap \tau$. Hence, cf(o.t.$(A \cap \tau)) = \omega_2$. Since $\tau = \sup(M \cap \omega_3)$ for some $M \in \mathcal{M}$, cf$(\tau) \leq \omega_1$ and so $A \cap \tau$ must be bounded below τ.

Proof of Claim 2. This is like case $M \in \lim(\mathcal{M})$ in the proof of 2.5 Lemma. Pick $M \in \mathcal{M}$ with $\pi, \tau \in A^M$. Then we have $\tau \in A^M \cap \pi$, $\gamma \leq \tau < \text{ssup}(A^M \cap \pi) = \text{ssup}(F_\pi^M)$ and $F_\pi^M \subseteq_{\text{end}} F_\pi$. Since $F_\pi^M \cap \text{ssup}(A^M \cap \pi) = F_\pi \cap \text{ssup}(A^M \cap \pi)$, we have $F_\pi^M \cap \gamma = F_\pi \cap \gamma$ and so $\gamma \leq \text{ssup}(A^M \cap \tau) = \text{ssup}(F_\tau^M)$. Since $F_\tau^M \subseteq_{\text{end}} F_\tau$, we have $F_\tau^M \cap \gamma = F_\tau \cap \gamma$. But by 2.5 Lemma, we have $F_\pi^M \cap \gamma = F_\tau^M \cap \gamma$. Hence, we have $F_\pi \cap \gamma = F_\tau \cap \gamma$.

Proof of Claim 3. For (1): Let i be a limit ordinal with $i < \omega_3$. Let $\pi \in A$ with $i = \Phi(\pi)$. Since i is limit, $A \cap \pi$ has no last elemenets and so F_π is cofinal in $A \cap \pi$. Then $\Phi[F_\pi]$ is a cofinal subset of i and so C_i is a closed and cofinal subset of i. We know that o.t.$(C_i) =$ o.t.$(\overline{\Phi[F_\pi]} \setminus \{i\}) =$ o.t.$(\Phi[F_\pi]) =$ o.t.$(F_\pi) \leq \omega_2$.

For (2): Suppose o.t.$(C_i) = \omega_2$. Then cf$(i) = \omega_2$.

For (3): Let $\Phi(\pi) = i$ and $\Phi(\tau) = j$. Hence, $\tau < \pi$ in A. We observe that $F_\pi \cap \tau$ is a cofinal subset of $A \cap \tau$. To do so let, $x \in A \cap \tau$. Then $\Phi(x) < \Phi(\tau) = j \in \overline{C_i} \cap i$. Then $\Phi(x) < \Phi(y) \in \Phi[F_\pi] \cap j$ and so $x < y < \tau$ for some $y \in F_\pi$. Hence, $A \cap \tau$ has no last elements and both F_τ and $F_\pi \cap \tau$ are cofinal subsets of $A \cap \tau$. Hence, $F_\tau = F_\pi \cap \tau$. Hence,

$$\Phi[F_\tau] = \Phi[F_\pi] \cap \Phi(\tau) = \Phi[F_\pi] \cap j.$$

Hence,

$$C_j = \overline{\Phi[F_\tau]} \setminus \{j\} = \overline{\Phi[F_\pi] \cap j} \setminus \{j\} = \overline{\Phi[F_\pi]} \cap j = C_i \cap j.$$

□

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